

REDUCTION AND NORMAL FORMS OF MATRIX PENCILS

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ABSTRACT. Matrix pencils, or pairs of matrices, may be used in a variety of applications. In particular, a pair of matrices (E, A) may be interpreted as the differential equation $Ex' + Ax = 0$. Such an equation is invariant by changes of variables, or linear combination of the equations. This change of variables or equations is associated to a group action. The invariants corresponding to this group action are well known, namely the Kronecker indices and divisors. Similarly, for another group action corresponding to the weak equivalence, a complete set of invariants is also known, among others the strangeness.

We show how to define those invariants in a directly invariant fashion, i.e. without using a basis or an extra Euclidean structure. To this end, we will define a reduction process which produces a new system out of the original one. The various invariants may then be defined from operators related to the repeated application of the reduction process. We then show the relation between the invariants and the reduced subspace dimensions, and the relation with the regular pencil condition. This is all done using invariant tools only.

Making special choices of basis then allows to construct the Kronecker canonical form. In a related manner, we construct the strangeness canonical form associated to weak equivalence.

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1. INTRODUCTION

1.1. Equivalence. The primary study of this paper is that of *pairs of matrices*, also called *matrix pencils*. In other words, we study pairs of operators (E, A) both acting from a finite dimensional vector spaces M to a finite dimensional vector space V .

A typical example we have in mind is the linear differential equation

$$(1) \quad E \frac{dx}{dt} + Ax = 0.$$

Such a model is clearly invariant by changes of variable, or by changing the order of the equations. More precisely, it is invariant by simultaneous *equivalence transformation* of the operators E and A . The corresponding equivalence relation is the following: two pairs of operators (E_1, A_1) and (E_2, A_2) will be considered equivalent if there exists invertible operators P and Q , operating on M and V respectively, such that

$$(2) \quad \begin{aligned} E_2 &= PE_1Q, \\ A_2 &= PA_1Q. \end{aligned}$$

This equivalence relation is associated to a group, which is simply $GL(M) \times GL(V)$. This is called *strong equivalence* in [1]. We are interested in properties which are invariant with respect to that group action on the matrix pencil. In other words, we are interested in quantities that label the orbit of the group action.

In fact, a complete set of invariants and a canonical form have been known since the works of [16] and [6]. Modern versions of those proofs may be found in [1, § XII.4] and in [2, § A.7]. The primary tool for obtaining those invariants is the Jordan canonical form. For that reason, those proofs are impossible to extend to nearby cases, for example to the infinite dimensional case, or to the parameter dependent case, not to mention the numerical difficulties associated with the computation of the Jordan canonical form.

As a result, alternative proof techniques were developed, most notably in [13] and [17]. Those authors observed indeed that using a Jordan canonical form is not suitable to compute the invariants other than the Jordan invariants, i.e., the Kronecker indices and the “infinite elementary divisors” [13]. The idea is to transform the pair of matrices into a form which exhibits all the invariants but is not a canonical form. Those forms are known under the names of generalized Schur-staircase form, or GUPTRI (Generalized Upper Triangular Form). We refer to [4, §4.1] and [8] for more references on those algorithms.

Our approach is similar, although with a shift of focus towards the underlying algebraic structures as opposed to the algorithmic aspects. In particular, we attempt to define the invariants from the dimensions of subspaces which are themselves invariants with respect to the equivalence relation at hand. The advantage of our approach is that a great deal of results are automatically independent of the choice of a basis, or any other structure (like a Euclidean structure).

1.2. Invariants. A matrix pencil, when considered as a differential equation (1), may be decomposed in an intrinsic ordinary differential equation, and an extra structure. We will call the invariants of the underlying ordinary differential equation the *dynamical* invariants, and we will call the remaining invariants the *non-dynamical invariants*. In the parlance of the Kronecker decomposition theorem as presented in [1], the dynamical invariants would be the finite elementary divisors (essentially a Jordan form), whereas the non-dynamical invariants would be the infinite elementary divisors along with the row and column minimal indices.

The dynamical invariants, i.e., the invariants of the intrinsic differential equations boil down to the Jordan invariant associated to similarity transformations, and are therefore of less interest to us. We will thus mostly focus on the non-dynamical invariants, which appear only when E is not invertible. Those invariants are well-known in control theory, and in the study of differential algebraic equations. In control theory, such invariants are the controllability and observability indices ([5, § 6.3]), for differential algebraic equations (DAE), in the case of a *regular pencil* (see subsection 3.8), the most used non-dynamical invariant is the *index* ([3, VII.1]).

Our goal is to define the non-dynamical invariants in a invariant manner, without any other structure than the linear algebraic structure.

1.3. Reduction. The crucial tool to the study of the invariants of a pencil is the concept of *reduction*, which we define precisely in Section 2.

This concept was gradually developed, under various names, or no name at all, first in [18] for the study of regular pencils, then in [17, §4] and [13] to prove the Kronecker decomposition theorem. It is also related to the *geometric reduction* of nonlinear implicit differential equations as described in [10] or [9]. In the linear case, those coincide with the observation reduction, as shown in [14]. It is also

equivalent to the algorithm of prolongation of ordinary differential equation in the formal theory of differential equations, as shown in [12].

The reduction procedure is an operation that, out of a pair of operators (E, A) , creates a new, smaller one (E', A') . “Smaller” is in the sense that the reduced operators E' and A' are restrictions of E and A on subspaces of M and V , defined by $V' := EM$ and $M' := A^{-1}V'$.

This process of reduction is iterated, producing systems $(E^{(k)}, A^{(k)})$ and subspaces $M^{(k)}$ and $V^{(k)}$. This process ultimately stops, and we will call the number of steps before it stops the *index*. When the process stops, the system which is produced, denoted by $(E^{(\infty)}, A^{(\infty)})$, is such that $E^{(\infty)}$ is surjective. After running the reduction algorithm once more on the dual of that reduced system, i.e., on $(E^{(\infty)*}, A^{(\infty)*})$, one obtains an isolated system $(E^{(\infty)*(\infty)}, A^{(\infty)*(\infty)})$ such that $E^{(\infty)*(\infty)}$ is now invertible.

At each step of the reduction, some information from the original system is lost. That information is encoded by integers called “defects”. Those defects are of three kinds: α , β^+ and β^- . The defect α_1 is defined as the dimension of the kernel of E , regarded as a quotient operator from M/M' to V'/V'' . The defect β_1^+ is defined as the dimension of the cokernel of A , regarded as a quotient operator from M/M' to V/V' . The iterated reduction then generates the sequences of defects α_k and β_k^+ . The defects β_k^- are defined as the β^+ defects of the system $(E^{(\infty)*}, A^{(\infty)*})$.

Using those subspaces, defined in an invariant manner, we are able to show the following facts:

- the operator E is invertible if and only if all the defects vanish
- the pair (E, A) is a regular pencil if and only if the β^+ and β^- defects vanish
- we show that the invariants defined in [7], like the strangeness, may also be defined directly in an invariant manner, i.e., without using any extra structure or basis

We also show that the defects and the system $(E^{(\infty)*(\infty)}, A^{(\infty)*(\infty)})$ completely characterize the equivalence class corresponding to the equivalence relation (2).

- the defects are related to the Kronecker indices
- the invariants defined in [7] may be used to construct a corresponding canonical form for weak equivalence: this connects the approaches of [17] and [7]
- using the relation with the Kronecker decomposition theorem, we show that the defects of the dual system (E^*, A^*) are related to those of (E, A) by switching the β^+ and β^- defects.

1.4. Outline. The layout of the paper is as follows.

In the first part, Section 2 and Section 3, we show how to derive the non-dynamical invariants. In Section 2 we define the reduction procedure. In Section 3 we define the defects of a system, and study their properties. In particular, we give an original proof of the relation between the property of a pencil to be regular, and the presence of some of the defects.

In the second part, we show that the invariants obtain in the first part, namely the defects, supplemented by a Jordan structure, are the only invariants of the pair of matrices with respect to equivalence. Most of the results in this part are already in [17] and [13]. In Section 4 we prove the basic lemmas needed to construct canonical forms. In Section 5, we show how to use those tools to construct a

canonical form with respect to weak equivalence. In Section 6 and Section 7 we show that the defects determine a complete canonical form. In Section 8 we study the relation with the existing Kronecker canonical form.

2. SYSTEM REDUCTION

2.1. Setting.

Definition 2.1. We will call a pair of linear operators (E, A) a **linear system**, or simply a **system**, if E and A have the same domain and codomain, both of finite dimension.

Given a system (E, A) , we will denote the common **domain** of E and A by $M_{(E,A)}$ and the common **codomain** of E and A by $V_{(E,A)}$, so a system (E, A) may be represented as

$$E, A : M_{(E,A)} \longrightarrow V_{(E,A)}.$$

2.2. Reduced spaces. The idea behind the reduction of a linear system (E, A) is to “disentangle” the spaces associated with the operators E and A . The strategy pursued is to try and make the operator E surjective, by successive reduction steps. In order to achieve this, we have to describe the lack of surjectivity of E , first independently of A , which leads to the definition of the subspace

$$V' := EM.$$

The next step is now to describe the lack of surjectivity of E , *with respect to* A , which we measure using the subspace

$$M' := A^{-1}V'.$$

Remark 2.2. Those definitions make sense when considering the differential equation

$$Ex' + Ax = 0.$$

Notice that any suitable initial condition for this equation must be in M' . If the initial condition is not in M' , there cannot be any solution stemming from that initial condition.

Let us put those definitions together:

Definition 2.3. Given a linear system (E, A) we define its **reduced codomain** $V'_{(E,A)}$ Reduced codomain as

$$V'_{(E,A)} := EM_{(E,A)},$$

and its **reduced domain** $M'_{(E,A)}$ as

$$M'_{(E,A)} := A^{-1}V'_{(E,A)} = \{x \in M_{(E,A)} : Ax \in V'_{(E,A)}\}.$$

Remark 2.4. We will often drop the dependency on the system (E, A) , and simply write M , M' , V and V' when the context is clear enough.

Remark 2.5. As explained in [14, §5.1], the reduction of Definition 2.3 corresponds to the non-linear reduction of general systems of differential equations with constraints. The study of differential equations is also the point of departure in [17].

Remark 2.6. One of the first occurrence of the definition of that subspace M' seems to be in [18, Lemma 2.1]. It is used to study systems which are regular pencils (see Definition 3.19).

Another explicit definition is to be found in [11, §7], although with a different purpose than ours, namely the study of linear, time-varying differential algebraic equations of index one.

2.3. System Reduction. The subspaces $M^{(k)}$ and $V^{(k)}$ defined in Definition 2.3 allow for defining a new system. This procedure will be called “reduction”.

Proposition 2.7. *Given a system (E, A) , the operators E' and A' are uniquely defined by the following commuting diagram.*

$$\begin{array}{ccc} M & \xrightarrow{E, A} & V \\ \uparrow & & \uparrow \\ M' & \xrightarrow{E', A'} & V' \end{array}$$

The vertical arrows are canonical injection from a subspace into the ambient space.

*The operators E' and A' build up a new system $(E, A)'$ which we call the **reduced system**, and is defined by*

$$(E, A)' := (E', A').$$

Proof. The proof rests on the observation that

$$EM'_{(E,A)} \subset V'_{(E,A)} \quad \text{and} \quad AM'_{(E,A)} \subset V'_{(E,A)}.$$

□

Remark 2.8. Consider the category which objects are vector spaces and arrows are systems as defined in Definition 2.1. The reduction operation, denoted by a prime, is an *endofunctor* in this category, i.e., a functor from that category to itself.

As we mentioned in the beginning of subsection 2.2, our goal is to obtain a reduced system such that E is surjective. It is only part of a general strategy to obtain a reduced system where E is invertible. It is therefore important that the reduction algorithm does not alter the injectivity of E . We observe that this is indeed the case.

Proposition 2.9. *If, in a system (E, A) , E is injective, then E' is also injective.*

Proof. It is a consequence of the observation that

$$\ker E' \subset \ker E.$$

□

The pendant of that observation is the equally simple observation regarding the kernel of the operator A with respect to the reduced space M' :

Proposition 2.10. *Given a system (E, A) , the null-space of A is included in $M'_{(E,A)}$, i.e.,*

$$\ker A \subset M'_{(E,A)}.$$

2.4. Iterated Reduction. We may iterate the reduction process described in subsection 2.3 on the new system $(E, A)'$. This leads to a sequence of systems $\{(E, A)^{(k)}\}_{k \in \mathbb{N}}$ which is defined recursively as follows.

Definition 2.11. The iterated of the reduction of a system (E, A) are defined recursively by

$$(E^{(k+1)}, A^{(k+1)}) := (E^{(k)}, A^{(k)})', \quad \forall k \geq 0,$$

and

$$(E^{(0)}, A^{(0)}) := (E, A).$$

We will make use of the straightforward notation, for $k \in \mathbb{N}$.

$$(3) \quad \begin{aligned} M_{(E, A)}^{(k)} &:= M_{(E, A)^{(k)}}, \\ V_{(E, A)}^{(k)} &:= V_{(E, A)^{(k)}}. \end{aligned}$$

The reduced operators $E^{(k)}$ and $A^{(k)}$ are essentially restrictions of the original operators E and A , so we may rewrite the definition of the iterated reduced subspaces $M^{(k)}$ and $V^{(k)}$.

Proposition 2.12. *For a system (E, A) the following assertions hold for any integer $k \geq 0$:*

$$\begin{aligned} \forall x \in M^{(k)} \quad E^{(k)}x &= Ex \quad A^{(k)}x = Ax, \\ V^{(k+1)} &= EM^{(k)}, \\ M^{(k+1)} &= \{x \in M^{(k)} : Ax \in V^{(k+1)}\}. \end{aligned}$$

Proof. The proof is a simple verification by induction on k . □

2.5. Totally Reduced Systems. As we shall notice in subsection 2.7, the repeated operation of reduction transforms a system into one which cannot be reduced anymore, or rather, for which the reduction does not create a new system. We call such systems “totally reduced”:

Definition 2.13. We will say that a system (E, A) is ***totally reduced*** if

$$(E, A)' = (E, A).$$

A practical characterisation of a totally reduced system is that $V' = V$. The verification is straightforward.

Proposition 2.14. *A system (E, A) is totally reduced if and only if*

$$V'_{(E, A)} = V_{(E, A)}.$$

2.6. Almost Reduced System.

Definition 2.15. We will say that a system (E, A) is ***almost reduced*** if

$$M'_{(E, A)} = M_{(E, A)}.$$

The chosen vocabulary is supported by the following facts:

- a totally reduced system is also almost reduced, which follows from Definition 2.3.
- a system which is almost reduced will be totally reduced at the next step of the reduction, since by Proposition 2.12: $V'' = EM' = EM = V'$.

Remark 2.16. In the situation of a reduced system which is almost but not totally reduced, the following subspace sequences

$$\begin{aligned} M^{(n+1)} = M^{(n)} &\subset \dots \subset M'' \subset M' \subset M \\ V^{(n+2)} = V^{(n+1)} &\subset V^{(n)} \subset \dots \subset V'' \subset V' \subset V \end{aligned}$$

would be produced.

A concrete example where this happens is when $A = 0$ and E is not surjective. It is clear that $M' = M$ but $V' \subsetneq V$. The corresponding system is thus almost reduced but not totally reduced.

2.7. Index. The reduction procedure produces decreasing sequences of subspaces. When both sequences stall, the system is totally reduced. The number of reduction steps needed to transform a system into a totally reduced one is called the *index* of the system (E, A) :

Definition 2.17. The smallest integer $n \in \mathbf{N}$ for which the system $(E^{(n)}, A^{(n)})$ is totally reduced is called the *index* of the system (E, A) .

We will use the following notation for the index of the system (E, A) :

$$\text{ind}_{(E,A)} := \min\{n \in \mathbf{N} : (E, A)^{(n+1)} = (E, A)^{(n)}\}.$$

Remark 2.18. The index is *always* a finite integer¹, and the reduced system (E', A') has an index dropped by one, i.e.,

$$\text{ind}_{(E,A)'} = \text{ind}_{(E,A)} - 1.$$

Those observations will be used repeatedly to prove statements by induction on the index (e.g., in Proposition 3.20, Theorem 6.1 and Theorem 8.2).

Remark 2.19. Using Proposition 2.14 we observe that

$$\text{ind}_{(E,A)} = \min\{n \in \mathbf{N} : V^{(n+1)} = V^{(n)}\}.$$

Remark 2.20. The index defined in Definition 2.17 is closely related to the geometric index defined in [10], [9] or [14, §5.1]. In fact, the geometric index would be the first integer n such that the system $(E, A)^{(n)}$ is *almost* reduced (Definition 2.15). As we shall see in Corollary 3.17 and Proposition 3.20, this minor difference is only relevant for singular pencils.

2.8. Totally Reduced System.

Definition 2.21. For a system (E, A) of index $n = \text{ind}_{(E,A)}$ we define the *totally reduced system* as

$$(E^{(\infty)}, A^{(\infty)}) := (E^{(n)}, A^{(n)}).$$

Remark 2.22. We could simply have defined, say $E^{(\infty)}$ by the limit of the sequence of operators $E^{(k)}$ (because this sequence eventually stalls), which explains the notation “ ∞ ”.

We pointed out in subsection 2.2 that the idea behind the reduction procedure was to lead to a system where E is surjective. The reduction algorithm indeed achieves this goal:

Proposition 2.23. *The totally reduced operator $E^{(\infty)}$ is surjective.*

¹as opposed to the differentiation index, which is infinite in the non-regular pencil case; see, e.g., [3, § VII.1].

Proof. The system $(E^{(\infty)}, A^{(\infty)})$ is totally reduced so we may use Proposition 2.14 to conclude that $E^{(\infty)}M^{(\infty)} = (V^{(\infty)})' = V^{(\infty)}$, so $E^{(\infty)}$ is surjective. \square

3. DEFECTS

3.1. Quotient Operators. At each step of the reduction some information is lost, by passing from the original system to the reduced one. We capture that information loss by two quotient operators defined on the quotient space M/M' .

Proposition 3.1. *The following commuting diagrams uniquely define the quotient operators $[A]$ and $[E]$ (the vertical arrows are the natural projections on a quotient space).*

$$\begin{array}{ccc} M & \xrightarrow{A} & V \\ \downarrow & & \downarrow \\ M/M' & \xrightarrow{[A]} & V/V' \end{array} \quad \begin{array}{ccc} M & \xrightarrow{E} & V' \\ \downarrow & & \downarrow \\ M/M' & \xrightarrow{[E]} & V'/V'' \end{array}$$

Moreover, $[A]$ is injective and $[E]$ is surjective.

Proof. The quotient operators $[A]$ and $[E]$ are well defined because $AM' \subset V'$ and $EM' \subset V''$ (since in fact, $EM' = V''$ by definition). $[E]$ is surjective because E is surjective onto V' by definition of V' . $[A]$ is injective since, by definition of M' ,

$$Ax \in V' \implies x \in M'.$$

\square

3.2. Constraint and Observation Defects. Since $[A]$ is injective and $[E]$ is surjective, the information stemming from those operators are to be collected in the cokernel of $[A]$ and the kernel of $[E]$. The dimension of those subspaces are important invariants of the system (E, A) which we now precisely define.

Definition 3.2. Let $[E]$ and $[A]$ be defined as in Proposition 3.1. We measure the lack of surjectivity of $[A]$ by the **first observation defect** $\beta_1^+(E, A)$, defined as

$$(4) \quad \beta_1^+(E, A) := \dim \operatorname{coker}[A]$$

and the lack of injectivity of $[E]$ by the **first constraint defect** $\alpha_1(E, A)$, defined as

$$(5) \quad \alpha_1(E, A) := \dim \ker[E].$$

Now we take advantage of the reduction procedure and define those defects recursively:

Definition 3.3. The **constraint defects** $\alpha_k(E, A)$ of a system (E, A) are defined for any integer $k \geq 1$ by

$$\alpha_k(E, A) := \alpha_1((E, A)^{(k-1)}).$$

Similarly, the **observation defects** $\beta_k^+(E, A)$ are defined for any integer $k \geq 1$ by

$$\beta_k^+(E, A) := \beta_1^+((E, A)^{(k-1)}).$$

3.3. Control Defects. There is another important kind of defect that will be needed. It is obtained by considering the dual of the totally reduced system obtained after repeated reductions. That totally reduced system $(E^{(\infty)}, A^{(\infty)})$ is such that $E^{(\infty)}$ is surjective, so $E^{(\infty)*}$ is injective. So what happens for a system (E, A) such that E is injective? It turns out that such a system has no constraint defects.

Proposition 3.4. *Given a system (E, A) , if E is injective, then the system has no constraint defects, i.e., for all integer $k \geq 1$, $\alpha_k(E, A) = 0$.*

Proof. The proof proceeds by induction on the index.

- (1) If the index is zero, then $M' = M$ and $V' = V$, so $\dim \ker[E] = 0$.
- (2) For a positive index, using Proposition 2.9 we may apply the induction hypothesis and deduce that $\alpha_k(E, A) = 0$ for $k \geq 2$.
- (3) Now if $x + M' \in \ker[E]$ then $Ex \in EM'$. Since E is injective, this means that $x \in M'$ and thus that $\ker[E] = 0$. We conclude that $\alpha_1(E, A) = 0$.

□

Let us introduce the notion of a dual system.

Notation 3.5. Given a system (E, A) we define the **dual system** $(E, A)^*$ by the pair of adjoint operators (E^*, A^*) , i.e.,

$$(E, A)^* := (E^*, A^*).$$

Proposition 3.6. *The dual $(E^{(\infty)}, A^{(\infty)})^*$ of a totally reduced system has no constraint defects, i.e.,*

$$\alpha(E^{(\infty)*}, A^{(\infty)*}) = 0.$$

Proof. According to Proposition 2.23, the operator $E^{(\infty)}$ is surjective, so $E^{(\infty)*}$ is injective, and we conclude using Proposition 3.4. □

This suggests that another set of defects is given by the observation defects of the dual of the totally reduced system $(E^{(\infty)}, A^{(\infty)})$.

Definition 3.7. Given a system (E, A) , we define the **control defects** $\beta_k^-(E, A)$ by

$$\beta_k^-(E, A) := \beta_k^+(E^{(\infty)*}, A^{(\infty)*}) \quad \forall k \geq 1.$$

3.4. Intrinsic Dynamical System. The reduction procedure may thus be used once to obtain a totally reduced system, and may then be applied again to the dual of that totally reduced system.

Starting with a system (E, A) , we may completely reduce it to obtain the system $(E^{(\infty)}, A^{(\infty)})$. The operator $E^{(\infty)*}$ is injective. The adjoint system $(E^{(\infty)*}, A^{(\infty)*})$ may be in turn completely reduced to obtain the system $(E^{(\infty)*(\infty)}, A^{(\infty)*(\infty)})$. Using Proposition 2.23 and Proposition 2.9, we obtain the following result.

Proposition 3.8. *The operator $E^{(\infty)*(\infty)}$ is invertible.*

Since the operator $E^{(\infty)*(\infty)}$ is invertible, its domain and co-domain have the same dimension. This dimension is the dimension of the intrinsic dynamics of the system.

Definition 3.9. The **dynamical dimension** δ of the system (E, A) is defined by the integer

$$\delta := \dim M^{(\infty)*(\infty)} = \dim V^{(\infty)*(\infty)}.$$

Remark 3.10. For a differential equation defined by the system (E, A) , the system

$$(E^{(\infty)*(\infty)*}, A^{(\infty)*(\infty)*})$$

corresponds to the underlying differential equation. In particular, the dynamical dimension δ determines the degrees of freedom for the choice of the initial condition.

3.5. Dimensions of the Subspaces. In order to study the relations existing between the defects and the various subspaces $M^{(k)}$ and $V^{(k)}$, we define the following spaces, which measure the difference of dimension between each successive reduction:

Definition 3.11. Recalling Definition 2.11, for any integer $k \geq 1$ we define the spaces

$$\Delta M^{(k)} := M^{(k-1)} / M^{(k)}$$

and

$$\Delta V^{(k)} := V^{(k-1)} / V^{(k)}.$$

By definition of the defects in Definition 3.2 and using Proposition 3.1, one obtains the relations

$$(6) \quad \begin{aligned} \dim \Delta M^{(k)} &= \dim \Delta V^{(k+1)} + \alpha_k, & \forall k \geq 1, \\ \dim \Delta V^{(k)} &= \dim \Delta M^{(k)} + \beta_k^+, & \forall k \geq 1, \end{aligned}$$

between the dimensions of the spaces defined in Definition 3.11 and the defects.

For any integer $k \geq 1$ this implies the inequalities

$$\dots \leq \dim \Delta M^{(k+1)} \leq \dim \Delta V^{(k+1)} \leq \dim \Delta M^{(k-1)} \leq \dim \Delta V^{(k)} \leq \dots$$

Remark 3.12. This is the same sequence of inequalities as in [17, 5.2].

In particular, the dimensions of the spaces $\Delta M^{(k)}$ and $\Delta V^{(k)}$ may be expressed using the constraint and observation defects.

Lemma 3.13. *For any integer $k \geq 1$, the dimensions of the spaces $\Delta M^{(k)}$ and $\Delta V^{(k)}$ are related to the defects by the identities*

$$\begin{aligned} \dim \Delta V^{(k)} &= \sum_{j \geq k} (\alpha_j + \beta_j^+), \\ \dim \Delta M^{(k)} &= \sum_{j \geq k} (\alpha_j + \beta_{j+1}^+). \end{aligned}$$

Proof. Those identities follow from an induction based on (6) and the observation that the integers $\dim \Delta V^{(k)}$ and $\dim \Delta M^{(k)}$ are zero when k is bigger than the index of the system. \square

Remark 3.14. As we shall see in Theorem 5.1, the quantity defined in [7] as the “strangeness” s turns out to be the integer

$$s = \dim \Delta V''.$$

Roughly speaking it expresses the number of constraints that, when differentiated, will help to reduce the system.

We may thus give the precise relation of the strangeness to the defects using Lemma 3.13, namely

$$s = \dim \Delta V'' = \sum_{k=2}^{\infty} \beta_k^+ + \sum_{k=2}^{\infty} \alpha_k.$$

The dimensions of the spaces M and V may also be expressed from the defects and the dynamical dimension δ (see Definition 3.9).

Proposition 3.15. *The dimensions of M , V , the defects α , β^+ and β^- and the dynamical dimension δ are related by the formulae*

$$\begin{aligned}\dim M &= \delta + \sum_{k \geq 1} k\alpha_k + \sum_{k \geq 1} k\beta_k^- + \sum_{k \geq 1} k\beta_{k+1}^+, \\ \dim V &= \delta + \sum_{k \geq 1} k\alpha_k + \sum_{k \geq 1} k\beta_k^+ + \sum_{k \geq 1} k\beta_{k+1}^-.\end{aligned}$$

Proof. First observe that since $\dim V^{(k)} = \dim V^{(k+1)} + \dim \Delta V^{(k+1)}$ and $\dim M^{(k)} = \dim M^{(k+1)} + \dim \Delta M^{(k+1)}$, we have

$$\dim M = \dim M^{(\infty)} + \sum_{k \geq 1} \dim \Delta M^{(k)} \quad \dim V = \dim V^{(\infty)} + \sum_{k \geq 1} \dim \Delta V^{(k)}.$$

Using Lemma 3.13 we obtain

$$\dim V = \dim V^{(\infty)} + \sum_{k \geq 1} k\alpha_k + \sum_{k \geq 1} k\beta_k^+,$$

and

$$\dim M = \dim M^{(\infty)} + \sum_{k \geq 1} k\alpha_k + \sum_{k \geq 1} k\beta_{k+1}^+.$$

Now using the observation of Proposition 3.6 that $\alpha(\mathbf{E}^{(\infty)*}, \mathbf{A}^{(\infty)*}) = 0$, along with Definition 3.7 of the defects β^- and Definition 3.9 of the dynamical dimension δ we readily obtain the result. \square

3.6. Relation with the Index. The index is, as expected, a non-dynamical invariant. More precisely, it is a function of the defects, as the following proposition shows:

Proposition 3.16. *The index $\text{ind}_{(\mathbf{E}, \mathbf{A})}$ (see Definition 2.17) of a linear system (\mathbf{E}, \mathbf{A}) is given by*

$$\text{ind}_{(\mathbf{E}, \mathbf{A})} = \min \{n \in \mathbf{N} : \forall k > n \quad \alpha_k(\mathbf{E}, \mathbf{A}) = 0 \quad \text{and} \quad \beta_k^+(\mathbf{E}, \mathbf{A}) = 0\}.$$

Proof. Following Remark 2.19, the index fulfills

$$\text{ind}_{(\mathbf{E}, \mathbf{A})} = \min_k \dim \Delta V^{(k+1)} = 0.$$

Using Lemma 3.13 we thus obtain

$$\dim \Delta V^{(k)} = 0 \iff \alpha_j + \beta_j^+ = 0 \quad \forall j \geq k+1,$$

which proves the claim. \square

In the case of a system without observation defects we obtain readily:

Corollary 3.17. *The index of a system (\mathbf{E}, \mathbf{A}) without observation defects (i.e., $\beta^+ = 0$) is the biggest index of non-zero constraint defects, i.e.,*

$$\text{ind}_{(\mathbf{E}, \mathbf{A})} = \min \{n \in \mathbf{N} : \forall k > n \quad \alpha_k(\mathbf{E}, \mathbf{A}) = 0\}.$$

backgroundcolor=green!40]add remark on index for DAEs?

3.7. Defects and Invertibility. The choice of the name “defect” may seem overly negative, but those integers really measure how far this system is from a system where E is invertible. This is the essence of the following proposition.

Proposition 3.18. *For a given system (E, A) the following statements are equivalent.*

- (i) *All the defects α , β^+ and β^- are zero.*
- (ii) *The operator E is invertible.*

Proof. E is surjective if and only if $\Delta V' = 0$. By Lemma 3.13, that is equivalent to $\alpha = \beta^+ = 0$. Since E is invertible if and only if both E and E^* are surjective, we obtain the result using Definition 3.7. \square

3.8. Regular Pencils. A *pencil* is a polynomial on a ring of matrices. Since we are interested in pairs of matrices, our attention is restricted to first order polynomials, and to the property of such a polynomial to be *regular*.

Definition 3.19. The system (E, A) is a **regular pencil** if there exists $\lambda \in \mathbf{C}$ such that $\lambda E + A$ is invertible.

There is a remarkable relation between the property of being regular and the defects:

Proposition 3.20. *The system (E, A) is a regular pencil if and only if all the defects β^+ and β^- are zero.*

We need first a lemma to understand how the pencil regularity property may be lost during the reduction.

Lemma 3.21. *The system (E, A) is a regular pencil if and only if both the following properties hold:*

- (i) $\beta_1^+(E, A) = 0$
- (ii) *The reduced system $(E, A)'$ is a regular pencil*

Proof. (1) Consider, for any $\lambda \in \mathbf{C}$, the operator S_λ defined by

$$S_\lambda := \lambda E + A.$$

S_λ can be decomposed into S'_λ and $[S_\lambda]$ according to the following commuting diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M/M' \longrightarrow 0 \\ & & \downarrow S'_\lambda & & \downarrow S_\lambda & & \downarrow [S_\lambda] \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V/V' \longrightarrow 0 \end{array}$$

Since both rows are exact sequence, out of the three operators S'_λ , S_λ and $[S_\lambda]$, if two of them are invertible then the third one is. One easy way to prove this fact² is by choosing bases in M and V which are compatible with the subspaces M' and V' . The operator S_λ is then represented by a block triangular matrix where the diagonal blocks are the matrices of S'_λ and $[S_\lambda]$. Now it is easy to check that if two of those three matrices are invertible, the third one is.

²This is a very general result that holds in other contexts as well, since one may also prove it by diagram chasing.

- (2) Notice that for any $\lambda \in \mathbf{C}$, $[S_\lambda] = [A]$ (the operator $[A]$ is defined in Proposition 3.1), so $[S_\lambda]$ is invertible if and only if $\beta_1^+ = 0$. As a result, we obtain the property

$$\beta_1^+ = 0 \implies [\forall \lambda \in \mathbf{C} \quad S_\lambda \text{ invertible} \iff S'_\lambda \text{ invertible}].$$

- (3) For any $\lambda \in \mathbf{C}$, the surjectivity of S_λ implies that of $[S_\lambda]$. Since $[S_\lambda]$ does not depend on λ , it means that if $[S_\lambda] = [A]$ is not surjective, then S_λ is not surjective for any $\lambda \in \mathbf{C}$. Now since, by definition, if $\beta_1^+ \neq 0$ then $[A]$ is not surjective, we conclude that

$$\beta_1^+ \neq 0 \implies \forall \lambda \in \mathbf{C} \quad S_\lambda \text{ not invertible.}$$

All the possibilities are covered and the claim is proved. \square

- Proof of Proposition 3.20.* (1) We first show by induction on the index that (E, A) is a regular pencil if and only if $\beta^+ = 0$ and $(E, A)^{(\infty)}$ is a regular pencil. It is easy to show using Lemma 3.21.
- (2) Now a system (E, A) is a regular pencil if and only if the dual system $(E, A)^*$ is a regular pencil, so we may apply on $(E^{(\infty)*}, A^{(\infty)*})$ the claim just proved. Because of Definition 3.7, we obtain that (E, A) is a regular pencil if and only if β^+ and β^- are zero, and $(E^{(\infty)*(\infty)}, A^{(\infty)*(\infty)})$ is a regular pencil.
- (3) Since, by Proposition 3.8, $E^{(\infty)*(\infty)}$ is invertible, the system $(E^{(\infty)*(\infty)}, A^{(\infty)*(\infty)})$ is a regular pencil, and the claim is proved. \square

4. COUPLING

4.1. Motivation: coupling spaces. In Section 2 we showed how to define *invariant* subspaces for the system (E, A) . “Invariant” means here that those subspaces are not arbitrarily chosen, they depend in a unique way from the system at hand.

In order to obtain a simple matrix representation of that system, we will need to choose supplementary spaces to the invariant subspaces M' and V' . In this section, we focus on such supplementary spaces for one reduction step only, and establish some results which will be needed in Section 6.

We first look at the case of supplementary subspaces to the subspace M' , i.e., subspaces $N' \subset M$ such that

$$M = M' \oplus N'.$$

The strategy is to try and choose N' in the same direction as the part of $\ker E$ that remains out of M' . First we define what this space is by decomposing the kernel of E in the part that is included in M' and some supplementary space. This is achieved by choosing any supplementary space K' such that

$$\ker E = (\ker E \cap M') \oplus K'.$$

Then since, by construction, $K' \cap M' = 0$ one may complete M' by choosing a supplementary space C' such that

$$M = M' \oplus C' \oplus K'.$$

We now define N' as

$$N' := C' \oplus K'.$$

The choice of C' will prove to be essential to obtain a complete decomposition of the system (E, A) . The tool to choose C' appropriately will be Lemma 4.2.

But notice now that no matter how we choose C' , the space K' roughly speaking corresponds to the variables that are *decoupled* from the rest of the system. They are sometimes called the *algebraic constraints*.

Example 4.1. Let us illustrate the previous remark by a trivial example. Consider the simple system

$$\begin{cases} x' = x \\ y = 0. \end{cases}$$

The variable y is *decoupled* from the rest of the system.

4.2. Coupling Lemma for E. We will assume that some coupling space W'' has already been chosen in the reduced system, and that will serve as a starting point for the choice of the coupling space at the present stage. More precisely, we assume that the reduced space V' is already decomposed as

$$V' = EM = EM' \oplus W''.$$

That decomposition allows to construct the coupling spaces in an optimal manner, in one subspace C' coupled with W'' , and complement with vectors in the null-space of E .

We state the result in a lemma, formulated outside the context of linear systems.

Lemma 4.2. *Assume that an operator E acting on a space M , and consider a subspace $M' \subset M$. For any subspace W'' such that $EM = EM' \oplus W''$ there exists subspaces K' and C' such that*

$$M = M' \oplus C' \oplus K'$$

and such that the sequence

$$0 \longrightarrow K' \longrightarrow K' \oplus C' \xrightarrow{E} W'' \longrightarrow 0$$

is exact. The exactness means here that $\ker E \cap (K' \oplus C') = K'$ and $E(K' \oplus C') = W''$.

Moreover, for any choice of basis in W'' one may choose a basis of C' such that its image by E is the basis in W'' (see Figure 1).

Proof. (1) Consider

$$\overline{C'} := E^{-1}W'' = \{x \in M : Ex \in W''\}.$$

Observe that $M' + \overline{C'} = M$, and $E\overline{C'} = W''$.

- (2) Pick $x \in \overline{C'} \cap M'$. It implies that $Ex \in W'' \cap EM'$, so $Ex = 0$, i.e., $x \in \ker E$. We conclude that $\overline{C'} \cap M' \subset \ker E$.
- (3) Choose C' such that $\overline{C'} = \ker E \oplus C'$. It follows from the previous observation that $C' \cap M' = 0$. This implies

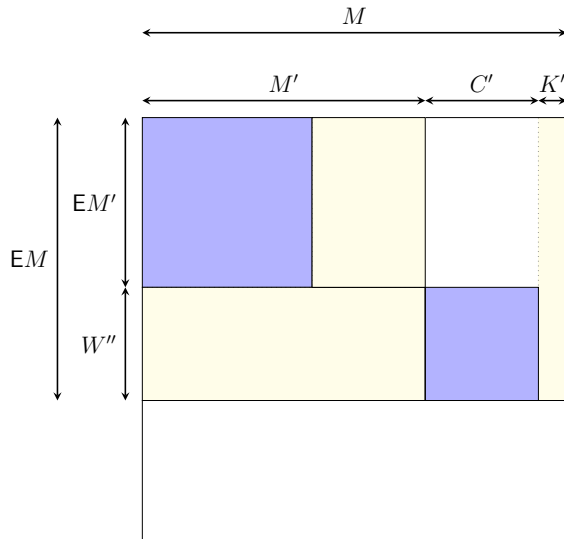
$$M = (M' + \ker E) \oplus C'.$$

- (4) Decompose further $\ker E$ as

$$\ker E = (\ker E \cap M') \oplus K'.$$

As a consequence, we obtain

$$M' + \ker E = M' \oplus K'$$



$$AN' \cap W' = A(N' \cap A^{-1}W').$$

With $W' = V'$ and since by Definition 2.3, $M' = A^{-1}V'$, we obtain

$$AN' \cap V' = A(N' \cap M')$$

from which the claim follows. \square

4.4. Coupling Lemma for Systems. We may now combine Lemma 4.2 and Lemma 4.3 and obtain a fundamental Lemma that decouples the operators E and A on supplementary spaces to M' and V' .

Lemma 4.4. *Suppose that there is a decomposition*

$$V' = V'' \oplus W'',$$

and that W'' is equipped with a basis.

Then there exists decompositions

$$M = M' \oplus N',$$

$$V = V' \oplus W',$$

and subspaces

$$C' \subset M \quad D' \subset V,$$

$$K' \subset M \quad Z' \subset V,$$

such that

$$(7) \quad N' = C' \oplus K',$$

$$(8) \quad W' = D' \oplus Z'.$$

Those subspaces are such that the following sequences are exact:

$$0 \longrightarrow K' \longrightarrow \underbrace{K' \oplus C'}_{N'} \xrightarrow{E} W'' \longrightarrow 0$$

$$0 \longrightarrow N' \xrightarrow{A} \underbrace{D' \oplus Z'}_{W'} \longrightarrow Z' \longrightarrow 0$$

and such that

$$AM \cap Z' = 0.$$

Moreover, one may choose basis in the subspaces C' , K' , D' and Z' such that the basis of D' is the image by A of the basis of N' , and the basis on W'' is the image by E of the basis of C' .

Proof. (1) By the assumption on W'' , we have

$$EM = V' = V'' \oplus W'' = EM' \oplus W''.$$

The subspace W'' is moreover equipped with a basis by the induction hypothesis.

(2) Appealing to Lemma 4.2 we obtain subspaces C' and K' such that

$$M = M' \oplus C' \oplus K'$$

with

$$EC' = W''$$

and

$$EK' = 0$$

and

$$\ker E \cap C' = 0.$$

Note that, given a basis in W'' we can choose a basis on C' such that E sends that basis on that of W'' .

Let us now define the subspace $N' \subset M$ by

$$N' := C' \oplus K'.$$

We choose an arbitrary basis of the space K' , and this provides us with a basis for the space N' .

- (3) Recall now that, according to Lemma 4.3,

$$AN' \cap EM = 0,$$

and by Proposition 2.10, the operator A sends the basis of the space N' to a set of independent vectors in the space V . We thus choose as a basis of AN' the image of the basis of N' by A .

- (4) Now choose a subspace $Z' \subset V$ such that

$$V = EM \oplus AN' \oplus Z'$$

and pick an arbitrary basis of that subspace. We define

$$D' := AN'$$

and

$$W' := D' \oplus Z'.$$

□

Remark 4.5. The dimensions of the spaces introduced in Theorem 6.1 are related to the dimensions of the spaces introduced in Definition 3.11, and to the defects (Definition 3.2). The relations are given by

$$\begin{aligned} \dim W' &= \dim \Delta V' & \dim Z' &= \alpha_1, \\ \dim N' &= \dim \Delta M' & \dim K' &= \beta_1^+. \end{aligned}$$

5. STRANGENESS

In order to illustrate the power of reduction, and to show an application of Lemma 4.4, we show an intermediate result. Instead of looking at the equivalence classes for the equivalence of matrices, that is, pairs of invertible operators acting on (E, A) as (PEQ, PAQ) , we look at the *weak* equivalence.

Weak equivalence is determined by another group, which elements consist of two invertible operators P and Q and an arbitrary operator R , acting on a system (E, A) as

$$(9) \quad (P, Q, R) \cdot (E, A) := (PEQ, P(ER + AQ)).$$

5.1. Weak equivalence group. The group operation corresponding to weak equivalence is given by

$$(P_2, Q_2, R_2) \cdot (P_1, Q_1, R_1) = (P_2 P_1, Q_1 Q_2, Q_1 R_2 + R_1 Q_2),$$

where P_1, P_2 are automorphisms of V , Q_1, Q_2 are automorphisms on M , and R_1, R_2 are arbitrary endomorphisms on M .

The identity is then

$$(I, I, 0),$$

and the inverse of an element (P, Q, R) is given by

$$(P, Q, R)^{-1} = (P^{-1}, Q^{-1}, -Q^{-1} R Q^{-1}).$$

Clearly, the elements of the form $(P, Q, 0)$ form a subgroup corresponding to the equivalence relation. Another subgroup is given by elements of the form $(\mathbb{I}, \mathbb{I}, R)$.

For the study of the orbits of the weak equivalence group, the identity

$$(10) \quad (P, Q, R) = (P, Q, 0) \cdot (I, I, R Q^{-1}) = (I, I, Q^{-1} R) \cdot (P, Q, 0)$$

shows that we may restrict our attention to one subgroup at a time.

5.1.1. Orbit Invariants. The orbits of the weak equivalence group action (9) were studied in [7], in which the authors exhibited a complete set of invariants. We give an alternative proof here, thereby shedding some light on the notion of *strangeness*.

Theorem 5.1. *A complete set of invariants for the group action (9) is given by*

- (1) $d := \dim V''$,
- (2) $a := \dim \ker[E] = \alpha_1$,
- (3) $s := \dim \Delta V''$.

The integer s is called “strangeness” in [7].

Proof. (1) First we have to check that the three integers are indeed invariants of the group action. Clearly, they are invariants by transformations of the form $(P, Q, 0)$, which are merely equivalent transformation.

Let us examine the case of a transformation

$$(\bar{E}, \bar{A}) = (\mathbb{I}, \mathbb{I}, R) \cdot (E, A) = (E, ER + A).$$

We have

$$\begin{aligned} \bar{V}' &= \bar{E}M = EM = V', \\ \bar{M}' &:= \{x : \bar{A}x \in \bar{E}M\} = M', \end{aligned}$$

so

$$(\bar{E}', \bar{A}') = (E, A)$$

and

$$V'' = \bar{E}'M' = EM' = V''.$$

Using (10), this shows that the spaces V' , V'' and M' are invariants of all of the weak equivalence group transformations.

As a result, the spaces $\Delta V''$ and the operator $[E]$ are also invariants, so the integers d , a and s are invariants.

- (2) Now we show that the integers d , a and s are the only invariants. In order to show that, we show that a system (E, A) is weakly equivalent to a canonical form that depends only on those three integers.

In order to achieve this, we decompose M and V using Lemma 4.4.

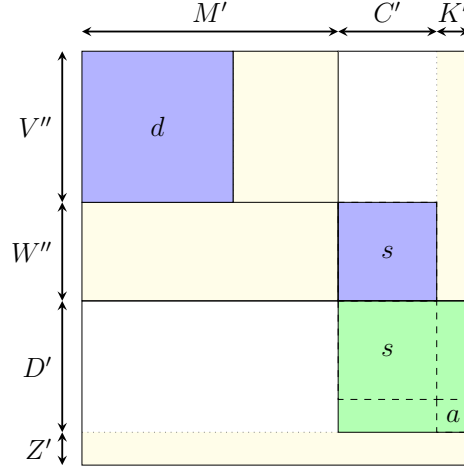


FIGURE 2. Canonical form of a matrix corresponding to the weak equivalence. The matrix E is represented in blue, whereas the matrix for A is represented in green. All such squares are identity matrices. The rest is filled out by zero entries.

- (a) Let us choose an arbitrary decomposition

$$V' = V'' \oplus W''.$$

We may now apply Lemma 4.4 to obtain spaces C' , K' , D' and Z' equipped with appropriate bases. Using Remark 4.5 we obtain $s = \dim W''$ and $a = \dim K'$.

- (b) Finally, define Π as a projector from M to M' along N' . Let F be a right inverse for E on $V' = EM$.

Define

$$R := -FA\Pi,$$

so $ER + A = 0$ on M' .

As a result, if we define the new system (\bar{E}, \bar{A}) by

$$(\bar{E}, \bar{A}) := (E, ER + A),$$

then the restriction of \bar{A} on M' is zero.

- (c) Now we may choose a basis of M' and of $V'' = EM'$ such that \bar{E} is represented by the identity matrix on M' .

This provides us with complete basis of M and V such that the matrices \bar{E} and \bar{A} take the form described in Figure 2.

□

6. DIRECT DECOMPOSITION

6.1. Decomposition Theorem. In Section 2 we showed how to define invariant subspaces $M^{(k)}$ and $V^{(k)}$ for the system (E, A) . In order to obtain a complete decomposition of the spaces M and V , it is necessary to construct subspaces that

bridge the gap between each invariant subspaces $M^{(k)}$ and $V^{(k)}$. More precisely we construct spaces $N^{(k)}$ and $W^{(k)}$ such that

$$M^{(k)} = M^{(k+1)} \oplus N^{(k+1)}, \quad V^{(k)} = V^{(k+1)} \oplus W^{(k+1)}.$$

In a sense, the subspaces $N^{(k)}$ and $W^{(k)}$ correspond to the spaces $\Delta M^{(k)}$ and $\Delta V^{(k)}$ respectively, defined in Definition 3.11.

The construction of those supplementary spaces proceeds backwards, in the direction opposite to the reduction. One must first totally reduce the system (E, A) . Assume that the index is n . One then chooses an arbitrary complementary space $W^{(n)}$ such that $V^{(n-1)} = V^{(n)} \oplus W^{(n)}$, and equip that space with an arbitrary basis. The rest is a repeated application of Lemma 4.4.

We now see how to choose those supplementary spaces $N^{(k)}$ and $W^{(k)}$ so that the operators E and A are *simultaneously* decomposed in an advantageous way.

Theorem 6.1. *Consider a system (E, A) .*

Recall the definitions of the subspaces $M^{(k)}$ and $V^{(k)}$ in (3).

For any integer $k \in \mathbf{N}$ there exists subspaces $N^{(k+1)} \subset M$ and $W^{(k+1)} \subset V$ such that

$$\begin{aligned} M^{(k)} &= M^{(k+1)} \oplus N^{(k+1)}, \\ V^{(k)} &= V^{(k+1)} \oplus W^{(k+1)}, \end{aligned}$$

and for any integer $k \geq 1$ there exists subspaces

$$\begin{aligned} C^{(k)} &\subset M & D^{(k)} &\subset V, \\ K^{(k)} &\subset M & Z^{(k)} &\subset V, \end{aligned}$$

such that

$$(11) \quad N^{(k)} = C^{(k)} \oplus K^{(k)},$$

$$(12) \quad W^{(k)} = D^{(k)} \oplus Z^{(k)}.$$

Those subspaces are such that for any integer $k \geq 1$, the following sequences are exact (see Figure 3):

$$0 \longrightarrow K^{(k)} \longrightarrow \underbrace{K^{(k)} \oplus C^{(k)}}_{N^{(k)}} \xrightarrow{E} W^{(k+1)} \longrightarrow 0$$

$$0 \longrightarrow N^{(k)} \xrightarrow{A} \underbrace{D^{(k)} \oplus Z^{(k)}}_{W^{(k)}} \longrightarrow Z^{(k)} \longrightarrow 0$$

and such that

$$AM^{(k-1)} \cap Z^{(k)} = 0.$$

Moreover, one may choose basis in the spaces $C^{(k)}$, $K^{(k)}$, $D^{(k)}$ and $Z^{(k)}$ such that the basis of $D^{(k)}$ is the image by A of the basis of $N^{(k)}$, and the basis on $W^{(k+1)}$ is the image by E of the basis of $C^{(k)}$.

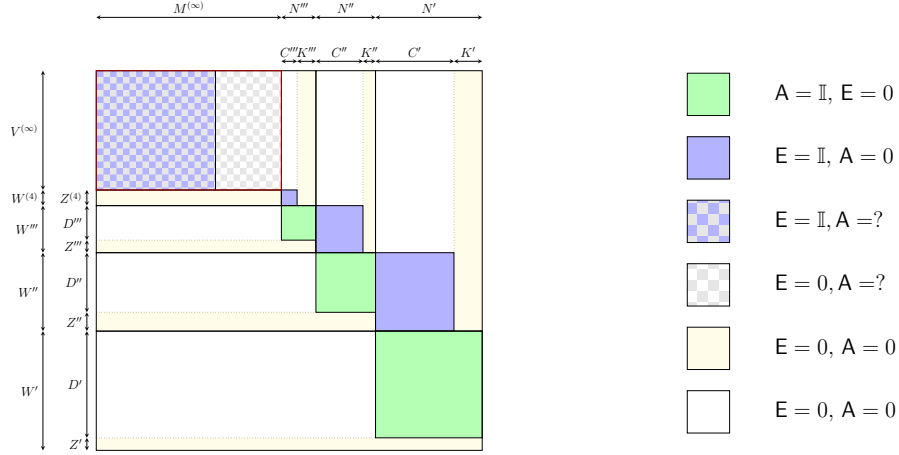


FIGURE 3. An illustration of the decomposition described in Theorem 6.1. Noticing that no matter what bases we choose in $M^{(\infty)}$ and $V^{(\infty)}$ the matrix is block diagonal (see Corollary 6.4), we may choose those bases in such a way that E is represented as the identity matrix on that block. Since we now that $E^{(\infty)}$ is surjective (Proposition 2.23), that identity block stretches to fill $V^{(\infty)}$.

Remark 6.2. For the reader averse to the language of exact sequences, the fact that the sequences of Theorem 6.1 are *exact* means in that case that

$$\begin{aligned} EC^{(k)} &= W^{(k+1)}, \\ EK^{(k)} &= 0, \\ AN^{(k)} &= D^{(k)}, \\ \ker A \cap N^{(k)} &= 0. \end{aligned}$$

Remark 6.3. In the same spirit as Remark 4.5, we notice the relation between the dimensions of the various subspaces introduced in Theorem 6.1, and the dimensions of the spaces defined in Definition 3.11, and to the defects (Definition 3.2). For any integer $k \geq 1$, the relations are given by

$$\begin{aligned} \dim W^{(k)} &= \dim \Delta V^{(k)} & \dim Z^{(k)} &= \alpha_k, \\ \dim N^{(k)} &= \dim \Delta M^{(k)} & \dim K^{(k)} &= \beta_k^+. \end{aligned}$$

Proof of Theorem 6.1. We proceed by induction on the index (see Figure 4). If the index is zero, all the spaces $M^{(k)}$ and $V^{(k)}$ are zero, and there is nothing to prove. Assume now that the statement holds for systems of index $n - 1$. Given a system (E, A) of index n , the reduced system (E', A') has index $n - 1$, so we may apply the induction hypothesis on that reduced system.

For clarity, let us denote

$$(\bar{E}, \bar{A}) := (E', A'), \quad \bar{M} = M', \quad \bar{V} = V'.$$

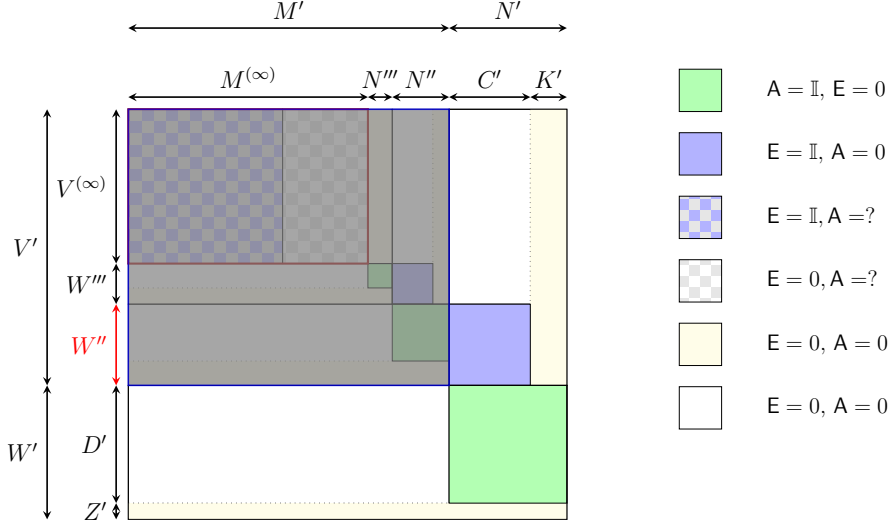


FIGURE 4. An illustration of Theorem 6.1 and Lemma 4.4 on an index three system. The grey shaded part pictures the previous step of the recursion. Starting with W'' , one constructs the space C' such that $EC' = W''$, and a subspace K' such that $K' \subset \ker E$ using Lemma 4.2, and defines $N' := C' \oplus K'$. One then constructs Z' such that $V = V' \oplus AN' \oplus Z'$. This in turn defines $W' := AN' \oplus Z'$.

The reduced system $(E, A)'$ consists of operators operating from M' to V' , so by the induction hypothesis we obtain a decomposition of the spaces \overline{M} and \overline{V} into subspaces $\overline{W}^{(k)}$, $\overline{N}^{(k)}$ as described in the statement of the theorem.

We have to shift the indices of all the spaces produced for the final statement to hold. For example, we define for any integer $k \geq 2$

$$W^{(k)} := \overline{W}^{(k-1)},$$

so we may write the decomposition of V' as

$$\overline{V} = V' = V^{(\infty)} \oplus W^{(n)} \oplus W^{(n-1)} \oplus \dots \oplus W''.$$

The reduced operators E' and A' being restrictions of E and A , the statements obtained from the induction hypothesis apply to the operators E and A .

Applying Lemma 4.4 yields the desired result.

6.2. Decomposition in invariant subspaces. A crucial consequence of Theorem 6.1 is that it provides us with decompositions of M and V such that E and A may be restricted on those subspaces:

Corollary 6.4. *Given the decomposition provided by Theorem 6.1, and defining \overline{M} and \overline{V} by*

$$\overline{M} := \bigoplus_k N^{(k)}, \quad \overline{V} := \bigoplus_k W^{(k)},$$

then, by construction,

$$M = M^{(\infty)} \oplus \overline{M}, \quad V = V^{(\infty)} \oplus \overline{V},$$

and we have

(i)

$$\mathbf{E}M^{(\infty)} = V^{(\infty)} \quad \mathbf{A}M^{(\infty)} \subset V^{(\infty)}$$

(ii)

$$\mathbf{E}\overline{M} \subset \overline{V} \quad \mathbf{A}\overline{M} \subset \overline{V}$$

Proof. The fact that $\mathbf{E}\overline{M} \subset \overline{V}$ and $\mathbf{A}\overline{M} \subset \overline{V}$ follows from

$$\mathbf{E}N^{(k)} \subset W^{(k+1)} \subset \overline{V}$$

and

$$\mathbf{A}N^{(k)} \subset W^{(k)} \subset \overline{V}.$$

□

7. DUAL DECOMPOSITION

7.1. Dual space decomposition. Assume that a finite dimensional vector space M is decomposed in a direct sum of subspaces, i.e.,

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_n.$$

This decomposition induces the dual space decomposition

$$(M_k)_* := \left(\bigoplus_{j \neq k} M_j \right)^\perp = \left\{ \varphi \in M^* : \langle \varphi, x \rangle = 0 \ \forall x \in \bigoplus_{j \neq k} M_j \right\}.$$

Although it is not reflected by the notation, it is clear that $(M_k)_*$ actually depends not only on M_k but on all the other spaces of the decomposition. It is a generalization of the notion of dual basis.

Assume further that M is equipped with a basis. We say that this basis is *compatible* with the decomposition if each subspace is the span of a subset of the basis.

If M is equipped with a basis \mathcal{B} compatible with a subspace decomposition, then the dual basis is compatible with the dual space decomposition.

Indeed, consider a subspace M_k of the decomposition. Since the basis is compatible with the decomposition, that subspace is spanned by a subset of the basis \mathcal{B} , say $\mathcal{S}_{M_k} \subset \mathcal{B}$, i.e.,

$$M_k = \text{span } \mathcal{S}_{M_k}.$$

The dual decomposition is such that the associated subspace $(M_k)_*$ is the span of the dual basis with the *same* subset \mathcal{S}_{M_k} , i.e.,

$$(M_k)_* = \text{span} \{ e^* : e \in \mathcal{S}_{M_k} \}.$$

Here the covector e^* is the element of the dual basis of \mathcal{B} corresponding to e , i.e., such that

$$\langle e^*, f \rangle = \begin{cases} 0 & \text{if } f \neq e \\ 1 & \text{if } f = e \end{cases}$$

Lemma 7.1. *Assume that A and B are subspaces of M that are part of a subspace decomposition of M , and that C is a subspace of V that is part of a subspace decomposition of V . Assume further that M and V are equipped with bases compatible with their decompositions. Take an operator S operating from M to V .*

The following two statements are equivalent:

(i) *The following sequence is exact:*

$$0 \longrightarrow A \longrightarrow A \oplus B \xrightarrow{S} C \longrightarrow 0$$

Moreover, the operator S sends the basis of B on the basis of C .

(ii) *The following “dual” sequence is exact:*

$$0 \longrightarrow C_* \xrightarrow{S^*} A_* \oplus B_* \longrightarrow A_* \longrightarrow 0$$

and

$$S^*V^* \cap A_* = 0.$$

Moreover, the operator S^ sends the basis of C_* on the basis of B_* .*

Proof. Denote the basis on M and V by $\mathcal{B}(M)$ and $\mathcal{B}(V)$ respectively. Clearly, for any $e \in \mathcal{B}(M)$ and $f \in \mathcal{B}(V)$, we have

$$\langle f^*, Se \rangle = \langle S^* f^*, e \rangle = \langle (e^*)^*, S^* f^* \rangle$$

where $(e^*)^*$ is the dual basis of the dual basis of \mathcal{B} .

The proof is now a simple verification by expressing each of the statements in terms of the bases. For example, $SA = 0$ may be written as

$$\langle f^*, Se \rangle = 0 \quad \forall e \in \mathcal{S}_A \quad f \in \mathcal{B}(V),$$

so one obtains

$$\langle (e^*)^*, S^* f^* \rangle = 0 \quad \forall e \in \mathcal{S}_A \quad f \in \mathcal{B}(V),$$

which means that $SA = 0 \iff S^*V^* \cap A_* = 0$. The other statements are verified in the same fashion. \square

7.2. Conjugate Decomposition. Consider a finite dimensional linear system (E, A) and its dual (E^*, A^*) . The corresponding domain and codomain are denoted by

$$\overline{M} := M_{(E, A)^*} = V^*,$$

$$\overline{V} := V_{(E, A)^*} = M^*.$$

By applying Theorem 6.1 on the dual system (E^*, A^*) one obtains a decomposition of \overline{M} and \overline{V} as

$$\overline{M} = \overline{M}^{(\infty)} \oplus \overline{K}' \oplus \overline{C}' \oplus \dots,$$

$$\overline{V} = \overline{V}^{(\infty)} \oplus \overline{Z}' \oplus \overline{D}' \oplus \dots.$$

Moreover, all those subspaces are equipped with a suitable basis. By choosing a basis for the spaces $\overline{M}^{(\infty)}$ and $\overline{V}^{(\infty)}$ we obtain compatible bases $\mathcal{B}(\overline{M})$ and $\mathcal{B}(\overline{V})$ of \overline{M} and \overline{V} respectively.

Theorem 7.2. *Consider the decompositions produced by Theorem 6.1 for the dual system $(E, A)^*$. The dual decompositions induce decompositions of the spaces M and V by the canonical isomorphism between a space and its bidual.*

For any integer $k \in \mathbf{N}$ we have

$$\begin{aligned} M_*^{(k)} &= M_*^{(k+1)} \oplus N_*^{(k+1)} \\ V_*^{(k)} &= V_*^{(k+1)} \oplus W_*^{(k+1)}, \end{aligned}$$

Those subspaces are such that the following sequences are exact (see Figure 3):

$$\begin{aligned} 0 \longrightarrow W_*^{(k+1)} &\xrightarrow{\mathbf{E}} \underbrace{K_*^{(k)} \oplus C_*^{(k)}}_{N_*^{(k)}} \longrightarrow K_*^{(k)} \longrightarrow 0 \\ 0 \longrightarrow Z_*^{(k)} &\longrightarrow \underbrace{D_*^{(k)} \oplus Z_*^{(k)}}_{W_*^{(k)}} \xrightarrow{\mathbf{A}} N_*^{(k)} \longrightarrow 0 \end{aligned}$$

Moreover, the basis of $C_*^{(k)}$, $K_*^{(k)}$, $D_*^{(k)}$ and $Z_*^{(k)}$ are such that the basis of $N_*^{(k)}$ is the image by \mathbf{A} of the basis of $D_*^{(k)}$, and the basis of $C_*^{(k)}$ is the image by \mathbf{E} of the basis of $W_*^{(k+1)}$.

Proof. It is a direct application of Lemma 7.1. \square

Remark 7.3. The exact sequences of Theorem 6.1 are the same as those of Theorem 7.2 but with flipped arrows. It just reflects how the block structure of a matrix is related to the block structure of the transposed matrix.

7.3. Second sweep of the decomposition. Remember that the constraint defects of the adjoint of the totally reduced system $(\mathbf{E}^{(\infty)*}, \mathbf{A}^{(\infty)*})$ are zero (Proposition 3.6). Along with Remark 6.3, we conclude that the corresponding subspaces $K^{(k)}$ produced in Theorem 6.1 are zero.

Now, using Corollary 6.4, we are in a position to use the decomposition of Theorem 6.1 for the dual system $(\mathbf{E}^{(\infty)*}, \mathbf{A}^{(\infty)*})$ and obtain a decomposition of $M^{(\infty)*}$ and $V^{(\infty)*}$.

Theorem 7.4. *In addition to the decomposition given by Theorem 6.1, the spaces $M^{(\infty)}$ and $V^{(\infty)}$ may now be decomposed as (see Figure 5):*

$$\begin{aligned} 0 \longrightarrow Z^{(k)} &\longrightarrow \underbrace{Z^{(k)} \oplus D^{(k)}}_{W^{(k)}} \xrightarrow{\mathbf{A}} N^{(k)} \longrightarrow 0 \\ 0 \longrightarrow W^{(k+1)} &\xrightarrow{\mathbf{E}} Z^{(k)} \longrightarrow 0 \end{aligned}$$

Proof. \square

Remark 7.5. The various defects defined in Definition 3.2 and Definition 3.7 may now be pictured clearly using the Theorem 7.4; see Figure 6.

8. KRONECKER INDICES

8.1. Basis Arrangement. In this section we prove a result on the basis obtained in Theorem 6.1, which will be useful to determine the relation with the Kronecker decomposition theorem.

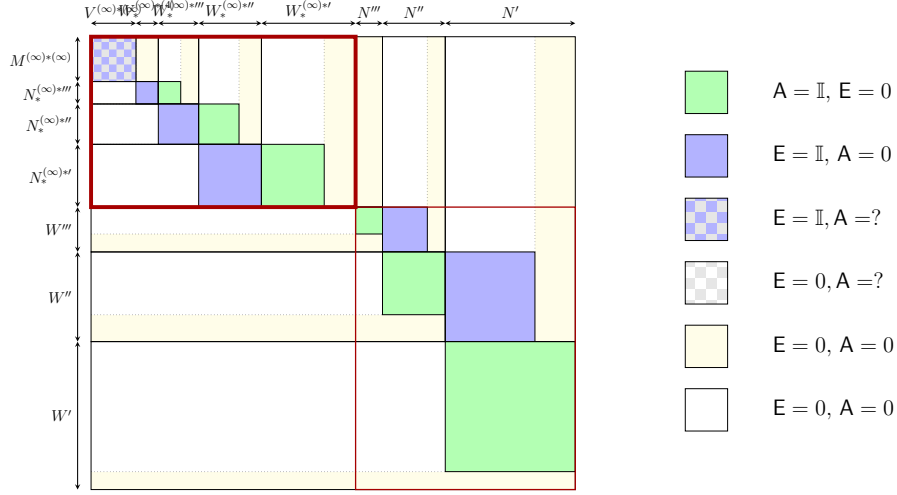


FIGURE 5. An illustration of the full decomposition of Theorem 7.4. The first decomposition leads to M'' and the corresponding space $V''' = EM''$, at which point the algorithm stalls. The second step consists in transposing the reduced operators $E^{(\infty)}$ and $A^{(\infty)}$ (indicated by the bold red frame on the figure), running the same algorithm, and transposing back again. The upper left checkered area denotes the identity for E , and a non specific matrix for A . Notice that this block is completely separated from the rest, so one may now reduce the A to Jordan blocks by a similarity transformation.

Definition 8.1. For $k \in \mathbf{N}$, $k \geq 1$, we define a N_k -**sequence** to be a sequence

$$m_j \in M \quad 1 \leq j \leq k$$

of k independent vectors in M , and a sequence

$$v_j \in V \quad 1 \leq j \leq k$$

of k independent vectors in V such that $Am_j = v_j$ for $1 \leq j \leq k$, $Em_j = v_{j-1}$ for $2 \leq j \leq k$, $Em_1 = 0$ and $v_k \notin \text{Im } E$, which is summarized in the following diagram.

$$0 \xleftarrow{E} m_1 \xrightarrow{A} v_1 \xleftarrow{E} \dots \xleftarrow{E} m_k \xrightarrow{A} v_k \notin \text{Im } E$$

Similarly, for $k \in \mathbf{N}$, $k \geq 1$, we define a L_k -**sequence** to be a sequence

$$m_j \in M \quad 1 \leq j \leq k-1$$

of $k-1$ independent vectors and a sequence

$$v_j \in V \quad 1 \leq j \leq k$$

of k independent vectors which fulfill the conditions summarized in the following diagram.

$$\text{Im } A \not\ni v_1 \xleftarrow{E} m_1 \xrightarrow{A} \dots \xleftarrow{E} m_{k-1} \xrightarrow{A} v_k \notin \text{Im } E$$

Theorem 8.2. *Theorem 6.1 produces bases such that there are α_k N_k -sequences, and β_k^+ L_k -sequences. Moreover, the end vectors \mathbf{v} constitute a basis of W' .*

- Proof.* (1) We proceed by induction on the index. Assume that the result holds for the reduced system (E', A') .
- (2) The basis of C' is precisely such that $E\mathbf{m}_{k+1} = \mathbf{v}_k$. Besides, the basis of AC' is chosen such that $\mathbf{v}_{n+1} = A\mathbf{m}_{k+1}$, so each L_k and N_k sequence is extended with two elements, meaning that they build now L_{k+1} and N_{k+1} sequences.
- (3) Each element \mathbf{m} of the basis of K' produces a new N_1 sequence $(\mathbf{m}, A\mathbf{v})$, since $E\mathbf{m} = 0$ and $A\mathbf{m} = \mathbf{v} \notin \text{Im } E$:

$$0 \xleftarrow{E} \mathbf{m} \xrightarrow{A} \mathbf{v} \notin \text{Im } E$$

The dimension of K' being α_1 , we produce α_1 such sequences.

- (4) Each element \mathbf{v} of the basis Z' qualifies as a L_0 sequence, since $\mathbf{v} \notin \text{Im } E$ and $\mathbf{v} \notin \text{Im } A$, so

$$\text{Im } A \not\ni \mathbf{v} \notin \text{Im } E.$$

Since the dimension of Z' is β_1^+ , we produce β_1^+ such sequences.

- (5) We conclude using Definition 3.3

□

8.2. Kronecker Decomposition. The Kronecker canonical form makes use of special blocks, each of which having a variant for the matrices E and A .

Definition 8.3. The rectangular ***bidagonal blocks*** $L_k^E E$ part of the Kronecker L-blocks and $L_k^A A$ part of the Kronecker L-blocks defined by

$$L_k^E := \left\{ \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \right\}_k, \quad L_k^A := \left\{ \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \right\}_k.$$

The ***nilpotent blocks*** $N_k^E E$ part of the nilpotent blocks and $N_k^A A$ part of the nilpotent blocks defined by

$$N_k^E := \left\{ \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} \right\}_k, \quad N_k^A := \left\{ \begin{bmatrix} 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ & & & 1 & 0 \\ & & & & 1 \end{bmatrix} \right\}_k.$$

Definition 8.4. A ***Kronecker decomposition*** of the system (E, A) is a choice of basis of M and V such that E and A are decomposed in blocks of the same size

$$E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \bar{E} \end{bmatrix}, \quad A = \begin{bmatrix} J & 0 \\ 0 & \bar{A} \end{bmatrix},$$

where J is a diagonal block of Jordan blocks, and \bar{E} and \bar{A} are in diagonal block form

$$\begin{aligned}\bar{E} &= \text{diag}(\mathbf{N}_{k_1}^E, \dots, \mathbf{N}_{k_m}^E, \mathbf{L}_{k_1}^E, \dots, \mathbf{L}_{k_p}^E, (\mathbf{L}_{k_1}^E)^T, \dots, (\mathbf{L}_{k_q}^E)^T), \\ \bar{A} &= \text{diag}(\mathbf{N}_{k_1}^A, \dots, \mathbf{N}_{k_m}^A, \mathbf{L}_{k_1}^A, \dots, \mathbf{L}_{k_p}^A, (\mathbf{L}_{k_1}^A)^T, \dots, (\mathbf{L}_{k_q}^A)^T),\end{aligned}$$

where the blocks of E and A have the same size.

Theorem 8.5. *A decomposition with defects α , β^+ , β^- , produces a Kronecker decomposition which for all integer $k \geq 1$ contains*

- α_k block of type \mathbf{N}_k ,
- β_k^+ blocks of type \mathbf{L}_k ,
- β_k^- blocks of type \mathbf{L}_k^T .

Proof. Recall the definition of L_k and N_k sequences in Definition 8.1. By regrouping the elements of an L_k sequence, one obtains a representation of E and A as a \mathbf{L}_k -block, and similarly, by regrouping the elements of a N_k -sequence, one obtains a \mathbf{N}_k block. Applying Theorem 8.2, and regrouping the basis elements stemming from the sequences L_k and N_k we obtain α_k \mathbf{N}_k -blocks and β_k^+ \mathbf{L}_k -blocks, for $k \geq 1$.

Now the basis on the sub-block $M^{(\infty)}$, $V^{(\infty)}$ are obtained by transposing the decomposition given by Theorem 6.1. Using the previous step and Proposition 3.6, we obtain β_k^- transpose of \mathbf{L}_k -blocks, for $k \geq 1$. \square

Remark 8.6. It is remarkable that the decomposition obtained in Theorem 7.4 produces basis vectors which are *the same* as for a Kronecker decomposition, only ordered differently. The necessary permutations may be visualised on Figure 6.

8.3. Conjugate Decomposition. We may now show the relation between the defects α , β^+ and β^- of a system (E, A) and the defects of the adjoint system (E^*, A^*) . It turns out that the constraint defects are the same and that the observation defects β^+ and the control defects β^- are just switched. This fact would have been very difficult to prove from the results of Section 3 alone, so we need the full power of Theorem 7.4 and of its consequence, Theorem 8.5.

Theorem 8.7. *The conjugate decomposition switches the defects β^+ and β^- , i.e., it produces the defects*

$$\begin{aligned}\alpha(E^*, A^*) &= \alpha(E, A), \\ \beta^+(E^*, A^*) &= \beta^-(E, A), \\ \beta^-(E^*, A^*) &= \beta^+(E, A).\end{aligned}$$

Proof. It is a consequence of Theorem 8.5, for when putting the system in Kronecker form Definition 8.4 and transposing, the system is still in Kronecker form, but the bidiagonal blocks \mathbf{L} and $(\mathbf{L})^T$ are switched. \square

8.4. Weierstraß decomposition. In the case of regular pencils (see subsection 3.8), the Kronecker decomposition is called the *Weierstraß decomposition* ([16, 1]) and is such that E and A take the matrix representation

$$E = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathbf{N} \end{bmatrix}, \quad A = \begin{bmatrix} \mathbf{C} & 0 \\ 0 & \mathbb{I} \end{bmatrix},$$

where \mathbf{C} may be in Jordan normal form and \mathbf{N} is a block diagonal matrix of blocks of type \mathbf{N}_k^E .

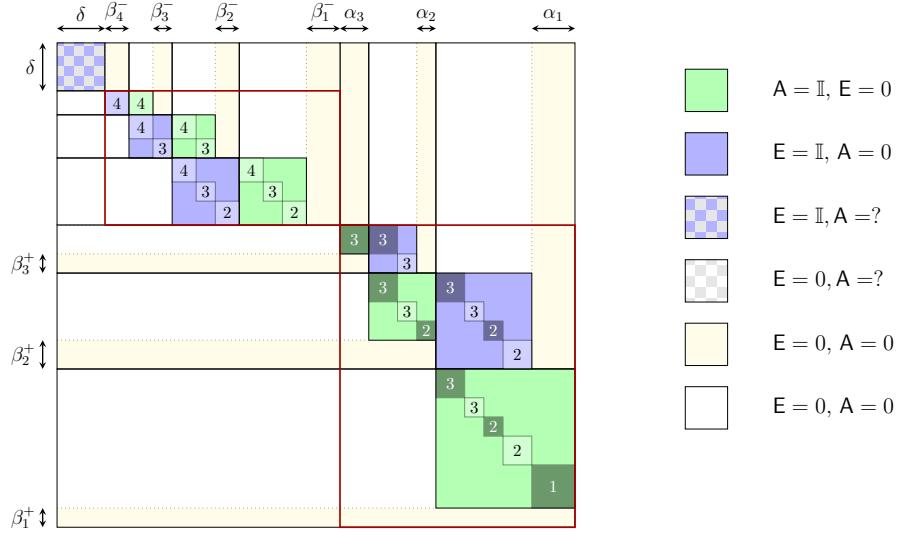


FIGURE 6. An illustration of the defects α , β^+ and β^- and of the Kronecker decomposition described in Theorem 8.5. The difference of size of the squares is exactly given by the defects α , β^+ and β^- . The dark squares bearing the number j represent all the nilpotent blocks N_j ; there are α_j such blocks. The light squares in the lower-right part bearing the number j represent the L-blocks L_j . There are β_j^+ such blocks. The light squares in the upper-left part bearing the number j represent the L-blocks L_j^T . There are β_j^- such blocks. This figure also allows to check the formulae of Proposition 3.15.

The matrix block NMatrix of nilpotent blocks is a diagonal block matrix

$$N = \text{diag}(N_{k_1}^E(0), N_{k_2}^E(0), \dots, N_{k_m}^E(0))$$

where the blocks $(N_{k_1}^E(0))$ are the nilpotent blocks defined in Definition 8.3.

Corollary 8.8. *The Weierstraß decomposition is such that for any integer $k \geq 1$ it contains α_k blocks N_k^E .*

Proof. It is just a special case of Theorem 8.5 using Proposition 3.20. \square

9. CONCLUSIONS

We have defined the notion of *defects* and have related them to existing concepts, such as the regular pencil condition, the dimension of the reduced subspaces, or the notion of strangeness. We also showed how the defects define a normal form, and how that normal form relates to the existing one of Kronecker.

Note that some results, as Theorem 8.7, would be difficult to prove without using the canonical form. Nevertheless, we tried to wring the most out of the invariant objects defined in Section 3.

The advantage of such an approach is that it is extensible to nearby cases such as the parameter dependent case, or the infinite dimensional case (see [15]).

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